The Logic of Geometric Proof

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Logical studies of diagrammatic reasoning—indeed, mathematical reasoning in general—are typically oriented towards proof-theory. The underlying idea is that a reasoning agent computes diagrammatic objects during the execution of a reasoning task. These diagrammatic objects, in turn, are assumed to be very much like sentences. The logician accordingly attempts to specify these diagrams in terms of a recursive syntax. Subsequently, he defines a relation $\vdash$ between sets of diagrams in terms of several rules of inference (or between sets of sentences and/or diagrams in case of so-called heterogeneous logics). Thus, diagrammatic reasoning is seen as being essentially a form of logical derivation. This proof-theoretic approach towards diagrammatic reasoning has been worked out in some detail, but only in a limited number of cases. For example, in case of reasoning with Venn diagrams and Euler diagrams (Shin [5] and Hammer [2]).

Nevertheless, when taken as a general approach towards the development of a logic of geometric proof, several problems can be pointed out (see below). These problems become manifest especially when we require that our logic should take account of certain cognitive and methodological features of geometric proof. In order to meet with these issues, we propose an alternative approach. The main difference is that, instead of proof-theory, ours is entirely oriented towards model-theory (cf. Barwise and Feferman [1]). We provide the first steps towards a mathematical formulation of our logic.

In order to bring to light the issues we have in mind, consider the following well-known theorem of elementary Euclidean plane geometry: the sum of the internal angles of a triangle is equal to two right angles. The following proof of this theorem is equally well-known. First, one constructs a triangle. Second, the base side of the triangle is extended. One sees that there arises an external angle adjacent to an internal one; together, these two angles are equal to two right angles. Third, the external angle is divided by constructing a line parallel to the opposite side. One sees that the two adjacent external angles that arise accordingly are respectively equal to the two opposite internal angles. This completes the proof.

The first object computed in the course of the proof described in the previous paragraph is referred to as a triangle. The second object can be referred to as a triangle *cum* extended base side, and so on. Thus, one obtains a series of objects such that any object in the series (except from the first) is always *built on* previous objects. In this sense, the proof proceeds constructively. The series of objects constructed approaches a "limit object." The informational content of this limit object can be described in terms of a statement of the theorem proved.

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As suggested earlier, it is often assumed that a reasoning agent computes diagrammatic objects in the course of a geometric proof. Upon closer inspection, however, the evidence that can be provided is rather shaky and often not very transparent. For example, a geometric proof typically goes together with the construction of a written diagram. However, on itself this can be hardly counted as evidence for the claim that the corresponding internal representations are also diagrammatic in nature. To this end, we need subtler methods. We will not enter into this vast and controversial territory here.

Larkin and Simon [4], p.106, have speculated that external diagrams play the same functional role as their corresponding internal representations. This makes us wonder about the precise role of external diagrams. It has been pointed out that in case of mechanical reasoning tasks, for example, an external diagram is often used as a form of external memory. One could then infer that the same function is fulfilled by the internal representation corresponding to it.

Note, however, that a memory resource in general not only serves to store information. Typically, one retrieves information from it as well. However, as is suggested by our example, geometric proofs proceed only by carrying out construction processes. Indeed, in case information would have been retrieved from an external diagram, then a geometric proof would certainly be considered as incorrect. In pure mathematics generally, "reading off" information from a diagram is not permissible when one's aim is to prove a theorem. As a result, when concrete diagrams are produced in the course of a geometric proof, their function as a medium of external memory is only limited. Though they may be used to store information, one does not subsequently retrieve this information in order to develop the proof, at least when this proof is to be carried out correctly. It would therefore seem that no information is retrieved from the corresponding internal representations as well.

What, then, *is* the function of the external diagrams that often accompany geometric proofs? Without elaborating deeply upon the issue, Larkin and Simon have also noted in passing that external diagrams sometimes represent information with substantially more detail than their corresponding internal representations (*ibid.*). We submit that this points towards an important cognitive function of external diagrams: they improve the quality of a geometric proof in that they make one see the truth of a theorem more clearly.

By way of summary, we provide two reasons why we cannot think of geometric proof as a logical derivation of a representation from others. First, as our example shows, the representations are constructively built on one another. Accordingly, a geometric proof is characterized by information *growth*. A logical derivation, in contrast, is always characterized by information *decrement*: the content of any derived representation (sentence or diagram) never goes beyond the content of the representation it is derived from. Second, as is suggested by our example, all representations are constructed. A logical derivation, in contrast, always starts from given representations (axioms or hypotheses).

When restricted to geometric proof, our considerations suggest the following (informal) semantic interpretation of "a proves φ": (i) a constructs a series of objects approaching a limit object $\mathfrak{A}$, (ii) the informational content of $\mathfrak{A}$ can be described in terms of φ. From an abstract logical point of view, what is computed is a series of first-order relational structures (possibly with constants) such that any structure in the series is "built on" previous structures. We can view this in terms of a productive directed system of relational structures. The direct limit of such a system satisfies the theorem.

Let I be a directed set, i.e., a set endowed with a pre-order $\leq$ such that $\forall i, j \in I : \exists k \in I : i, j \leq k$. For any $i \in I$, let Σ_i be a signature and let $\mathfrak{A}_i$ be a relational Σ_i-structure. Denote the universe of $\mathfrak{A}_i$ as A_i (which is nonempty). Whenever $i \leq j$, we assume that we have a mapping $\varphi_{ij} : A_i \to A_j$. We call the family $\{\mathfrak{A}_i, \varphi_{ij}\}$ a *productive direct system* provided that:

(1) $\varphi_{ii} = 1$ $(i \in I)$ and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}, i \leq j \leq k$;
(2) for any relation symbol $R \in \bigcup \Sigma_i$ there exists $i \in I$ such that $R \in \Sigma_j$ whenever $i \leq j$; similarly, for any constant $c \in \bigcup \Sigma_i$ there exists $i \in I$ such that $c \in \Sigma_j$ whenever $i \leq j$;
(3) given $\varphi_{ij} : A_i \to A_j$ $(i \leq j)$, then for every n-ary $R \in \Sigma_i \cap \Sigma_j$,

$$\{\varphi_{ij}(x_1, \ldots, x_n) : (x_1, \ldots x_n) \in R^{\mathfrak{A}_i}\} \subseteq R^{\mathfrak{A}_j}$$

and for every constant $c \in \Sigma_i \cap \Sigma_j$ we have $\varphi_{ij}(c^{\mathfrak{A}_i}) = c^{\mathfrak{A}_j}$.

It is because of condition (3) that the system is called productive.

Let $A = \prod A_i$. Where $a, b \in A$, let $a \equiv b$ iff $\exists k \in I : \forall i \geq k : a(i) = b(i)$. Then $\equiv$ is an equivalence relation on A. Denote the equivalence class of $a \in A$ in $A/\equiv$ as $\overline{a}$. Let $p : A \to A/\equiv$ be the projection. The restriction $\pi_i = p|A_i$ is called the canonical map of A_i into $A/\equiv$. Clearly, $\bigcup \pi_i(A_i) = A/\equiv$. It follows that $A/\equiv$ is nonempty.

Define a model $\mathfrak{A}$ with universe $A/\equiv$ as follows. For every n-ary relation symbol $R \in \bigcup \Sigma_i$ there exists $i \in I$ such that $(\overline{a}_1, \ldots, \overline{a}_n) \in R^{\mathfrak{A}}$ iff for every $j \geq i, (a_1(j), \ldots a_n(j)) \in R^{\mathfrak{A}_j}$. Similarly, for every constant $c \in \bigcup \Sigma_i$ there exists $i \in I$ such that $c^{\mathfrak{A}} = \overline{a}$ iff for every $j \geq i$, $c^{\mathfrak{A}_j} = a(j)$. This interpretation is well-defined. The model $\mathfrak{A}$ is called the *direct limit* of the system $\{\mathfrak{A}_i, \varphi_{ij}\}$ and is denoted as $\varinjlim \mathfrak{A}_i$.

In our definition of a productive direct system we could have opted for a fixed signature for all the structures instead of letting each structure have its own signature. However, from a computational point of view, it is preferable to have a relation symbol or a constant available only when one needs it. Furthermore, once introduced, a relation symbol or constant should remain available from then on. This is what motivates condition (2).

Often, Σ_i is a subset of Σ_j and φ_{ij} is injective whenever $i \leq j$. In such a case, it follows that $\mathfrak{A}_i$ is embedded in the Σ_i-reduct of $\mathfrak{A}_j$. From the assumption that the φ_{ij} are injective, it follows that the π_i are also injective. Hence, each model $\mathfrak{A}_i$ embeds in $\mathfrak{A}$.

Suppose we also assume that the index set I is well-ordered, so that $\{\mathfrak{A}_i\}$ is a chain. Recall that $a \equiv b$ iff a and b are both identical to a constant value after some point. Denote this constant value as z. Then $f : A \to \bigcup A_i$ given by

$f(\overline{a}) = z$ is well-defined. Furthermore, f is an isomorphism between $\mathfrak{A}$ and the union of the chain $\{\mathfrak{A}_i\}$.

In case of our earlier example, the limit object was the last object in the series. Note, however, that not every geometric proof proceeds by constructing a series of objects having a last object. For example, some proofs appear to proceed by the construction of an infinite series of objects, e.g., when one (geometrically) proves the existence of space filling curves by way of an appropriate infinite sequence of curves uniformly converging to a limit curve. Upon closer inspection, however, not all the terms in the sequence are actually produced. For example, one often constructs the first two or three terms in the sequence and accordingly sees the possibility of an infinite sequence of objects (e.g., Kuratowski [3], p.222-4). Thus, our approach describes possible proofs, and these can be infinite. Alternatively, one can specify the sequence analytically (e.g., Sierpiński [6]). In this case, however the proof seems to lose its peculiar geometric character and therefore becomes considerably less relevant to consider in the present context.

Logically speaking, an agent computes a model by employing axioms, definitions and theorems of an underlying theory. For example, a triangle is computed by employing the definition of a triangle (defining a triangle as, for example, a certain system of line segments). One typically computes a triangle by constructing a *minimal* model of the definition of a triangle. Intuitively, the model is minimal in the sense that one ignores any relation (or constant) not cited in this definition, e.g., the length of the sides (see also below). Subsequently, one extends the base side of the triangle by employing further axioms, theorems, and/or definitions, and so on. Accordingly, our attention is drawn to a language. The sentences (or sets thereof) of this language are interpreted as procedures to the effect of constructing certain models, generally by building on other models.

Considering our running example, the construction of a model may in very broad outlines proceed as follows. First note that one typically announces the construction of a triangle verbally by "let ABC be a triangle." We may imagine that a signature is introduced consisting of the constants AB, BC and AC. We can then consider the Herbrand universe corresponding to this signature and interpret its elements as the respective sides of the triangle. In such a way, one builds a model of the definition of a triangle. One subsequently employs other sentences in order to carry out further constructions. For example, one announces the extension of the base side of the triangle by saying "extend the base side AB to a point D" and then uses the relevant axioms and/or theorems. We may imagine that the constant BD is added to the signature and that the universe of the model is extended accordingly so as to obtain a Herbrand universe corresponding to the new signature. Thus, a second model is built. Note that the first model is a submodel of a reduct of the second. A detailed specification of the entire computation requires some work. Unfortunately, we cannot treat the matter here.

As said, mathematician proving the theorem typically writes down the abstract models in terms of concrete diagrams. The object in its written form has a lot more determinations when compared with the corresponding model, which is abstract in turn. For example, the sides of the diagram will have a determinate length, the angles will have a determinate size etc. Be that as it may, in order to prove the theorem one merely has to run a number of procedures associated with sentences of an underlying theory. Since one only takes account of these processes, the irrelevant properties and relations are disregarded accordingly. When the proof is taken as a process, the objects computed are partial objects.

We believe that our model-theoretic approach can be applied to proofs in algebra too. In this case, however, the models considered are algebraic structures instead of relational ones.

References

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